

Properties & Theorems of Determinants

$$A = [a_{ij}]_{3 \times 3}$$

$$(1) \quad a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13} = |A|$$

$$a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} = |A|$$

Sum of products of elements of any row (or column) of A with their cofactors is determinant of A ($|A|$)

$$(2) \quad a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23} = 0$$

$$a_{13} C_{11} + a_{23} C_{21} + a_{33} C_{31} = 0$$

Sum of products of elements of any row (or column) of A with cofactors of any other row (or column) is 0

$$(3) \quad |A| = |A^T| \quad \text{ie.} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$$

$$(4) \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix} = - \begin{vmatrix} c & d \\ a & b \end{vmatrix} \quad \begin{vmatrix} R_1 \\ R_2 \\ R_3 \end{vmatrix} = - \begin{vmatrix} R_3 \\ R_2 \\ R_1 \end{vmatrix}$$

$R_1 \leftrightarrow R_2$ $R_1 \leftrightarrow R_3$

$$(5) \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \end{vmatrix} = \begin{vmatrix} a_{31} & a_{32} & a_{33} \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{vmatrix}$$

Row passed over even number of rows

$$(6) \quad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \quad \text{Any 2 rows (or columns) identical / proportional}$$

$$(7) \begin{vmatrix} a+b & c+d \\ e & f \end{vmatrix} = \begin{vmatrix} a & c \\ e & f \end{vmatrix} + \begin{vmatrix} b & d \\ e & f \end{vmatrix}$$

$$\begin{vmatrix} a+b & c+d & e+f \\ g & h & i \\ j & k & l \end{vmatrix} = \begin{vmatrix} a & c & e \\ g & h & i \\ j & k & l \end{vmatrix} + \begin{vmatrix} b & d & f \\ g & h & i \\ j & k & l \end{vmatrix}$$

$$(8) \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a+kc & b+kd \\ c & d \end{vmatrix}$$

$R_1 \rightarrow R_1 + kR_2$

$$(9) \begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

$$(10) \begin{vmatrix} la & lb \\ mc & md \end{vmatrix} = l \begin{vmatrix} a & b \\ mc & md \end{vmatrix} = lm \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\begin{vmatrix} la & lb \\ lc & ld \end{vmatrix} = l^2 \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$|kA| = k^n |A|, \quad |-A| = (-1)^n |A|$$

$$(10) \Delta(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & l(x) \end{vmatrix}, \quad f(a)=0, g(a)=0 \Rightarrow \text{if } f \text{ \& } g \text{ are polynomials}$$

then $(x-a)$ is factor of $\Delta(x)$.

$$(11) \Delta(x) = \begin{vmatrix} f(x) & g(x) \\ h(x) & l(x) \end{vmatrix}, \quad \Delta'(x) = \begin{vmatrix} f'(x) & g'(x) \\ h(x) & l(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ h'(x) & l'(x) \end{vmatrix}$$

$$\Delta(x) = \begin{vmatrix} R_1(x) \\ R_2(x) \\ R_3(x) \end{vmatrix}, \quad \Delta'(x) = \begin{vmatrix} R_1'(x) \\ R_2(x) \\ R_3(x) \end{vmatrix} + \begin{vmatrix} R_1(x) \\ R_2'(x) \\ R_3(x) \end{vmatrix} + \begin{vmatrix} R_1(x) \\ R_2(x) \\ R_3'(x) \end{vmatrix}$$

$$(12) \text{ Standard Determinants (to learn)}$$

$$A) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a) \quad B) \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

$$C) \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)((a-b)^2 + (b-c)^2 + (c-a)^2)$$

(13) $|AB| = |A| |B|$

A & B are square matrices of same order

(14)
$$\begin{vmatrix} a_1 l_1 + a_2 l_2 & a_1 m_1 + a_2 m_2 \\ b_1 l_1 + b_2 l_2 & b_1 m_1 + b_2 m_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 \\ l_2 & m_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \times \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}$$

Method of Solving 3 linear equations in 3 unknowns

$$\left. \begin{aligned} a_1 x + b_1 y + c_1 z &= d_1 \\ a_2 x + b_2 y + c_2 z &= d_2 \\ a_3 x + b_3 y + c_3 z &= d_3 \end{aligned} \right\} \text{(3 planes)}$$

Cross multiplication Method / Cramers Rule

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Matrix Method

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

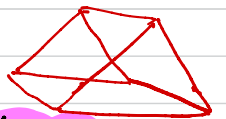
system given by
 $AX = B$

Case 1: $D \neq 0$ Unique solution given by $\left(\frac{D_1}{D}, \frac{D_2}{D}, \frac{D_3}{D} \right)$ or $x = A^{-1}B$
or $|A| \neq 0$ (3 planes intersect at a point)

Case 2: $D = 0, D_1 \neq 0 \text{ or } D_2 \neq 0 \text{ or } D_3 \neq 0$ No solution / System Inconsistent
or $|A| = 0, \text{Adj}(A) \cdot B \neq 0$

Case 3: $D = 0, D_1 = D_2 = D_3 = 0$ Infinitely many solutions
or $|A| = 0, \text{Adj}(A) \cdot B = 0$ (provided no two planes are parallel)

Infinite Solutions: Solve 2 equations for x & y in terms of z



Homogeneous System

$$a_{1x} + b_1y + c_1z = 0$$

$$a_{2x} + b_2y + c_2z = 0$$

$$a_{3x} + b_3y + c_3z = 0$$

3 planes passing through (0,0,0)

Case 1 $D \neq 0$, Unique solution (0,0,0) /
Trivial solution /
only zero solution

Case 2 $D = 0$, Infinitely many solutions /
Non Trivial solution /
Non zero solution.

Matrices

Definitions

→ Row matrix $[a_{11} \ a_{12} \ \dots \ a_{1n}]_{1 \times n}$

→ Column matrix $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}_{m \times 1}$

→ Null matrix $A = [0]_{m \times n}$ all elements 0

→ Square matrix $A = [a_{ij}]_{n \times n}$ or $A_{n \times n}$

no. of rows = no. of columns

→ Diagonal matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0 \forall i \neq j$

→ Scalar matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0 \forall i \neq j$
 $a_{ii} = k$

→ Unit / identity matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0 \forall i \neq j$
 $a_{ii} = 1$

→ Upper Triangular matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0 \forall i > j$

→ Lower Triangular matrix $A = [a_{ij}]_{n \times n}$, $a_{ij} = 0 \forall i < j$

→ Trace or Sum $A = [a_{ij}]_{n \times n}$ $\text{Tr}(A) = \sum_{i=1}^n a_{ii}$

→ Comparable matrices $A_{m \times n}$, $B_{m \times n}$ same order
(can be added / subtracted)

→ Equality of matrices $A = B$ if (a) Both have same order
" $[a_{ij}]_{m \times n} = [b_{ij}]_{m \times n}$
(b) $a_{ij} = b_{ij}$

→ Transpose of matrix A^T or $A' = [a_{ji}]_{n \times m}$
 $A = [a_{ij}]_{m \times n}$ Row $\rightarrow$ Columns

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, A^T = A' = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

→ Singular matrix $|A| = 0 \rightarrow$ Non Singular
 $|A| \neq 0$

→ Symmetric matrix

$[n]$, $\begin{bmatrix} x & a \\ a & y \end{bmatrix}$, $\begin{bmatrix} x & a & b \\ a & y & c \\ b & c & z \end{bmatrix}$ $A^T = A$ $a_{ij} = a_{ji}$

→ Skew Symmetric matrix

$[0]$, $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$ $A^T = -A$ $a_{ij} = -a_{ji} \Rightarrow a_{ii} = 0$

→ Idempotent matrix A_n if

$A^2 = A$
 $A^{k+1} = A$

→ Periodic

→ Involutory matrix A_n if

$A^2 = I$

→ Nilpotent matrix A_n if

$A^k = 0$ (null matrix)
& $A^{k-1} \neq 0$

→ Orthogonal matrix A_n

$A \cdot A^T = A^T \cdot A = I$

Properties of Addition / Multiplication /
Symmetric / Skew Symmetric / Transpose

① $A + B = B + A$
 $A + A = 2A$

⑤ $A \cdot B \neq B \cdot A$
 $A \cdot A = A^2$, $A^3 = A^2 \cdot A = A \cdot A \cdot A$

② $A + (B + C) = (A + B) + C$

⑥ $A \cdot (B \cdot C) = (A \cdot B) \cdot C$

③ $A + 0 = A$

⑦ $A \cdot I = I \cdot A = A$

④ $A + (C - A) = C$

⑧ $A \cdot A^T = A^T \cdot A = I$

(9) $A \cdot (B \pm C) = A \cdot B \pm A \cdot C$
 $(B \pm C) \cdot A = B \cdot A \pm C \cdot A$

(10) $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$
 $(A+B)(A-B) = A^2 - A \cdot B + B \cdot A - B^2$

(11) $(A^T)^T = A$ (12) $(A \pm B)^T = A^T \pm B^T$

(13) $(A \cdot B)^T = B^T A^T$ (14) $(kA)^T = k(A^T)$

$(ABC)^T = C^T B^T A^T$

(15) $(A^T)^{-1} = (A^{-1})^T$

(16) For any square matrix A

$\left. \begin{matrix} A + A^T \\ A \cdot A^T \\ A^T \cdot A \end{matrix} \right\} \rightarrow \text{Symmetric matrices}$

$A - A^T$ Skew Symmetric

(17) $A \rightarrow \text{Symmetric}, A^T \rightarrow \text{Symmetric}$
 $A^n \rightarrow \text{Symmetric}$
 $A^{-1} \rightarrow \text{Symmetric}$

$A \rightarrow \text{Skew Symmetric}$
 n even
 n odd

$A^n \rightarrow \text{Symmetric}$

$A^n \rightarrow \text{Skew Symmetric}$

(18) $A \rightarrow \text{Skew symmetric of odd order}, |A| = 0$
 $\rightarrow \text{skew symmetric of even order}, |A| \text{ is perfect square.}$
 (with integer elements)

(19) $A = \underbrace{A + A^T}_2 + \underbrace{A - A^T}_2$
Symmetric Skew symmetric

(20) $AB = 0 \nRightarrow A = 0 \text{ or } B = 0$
 $AB = AC \nRightarrow A = 0 \text{ or } B = C$

Adjoint of square matrix A

$\text{Adj}(A) = \text{Transpose of Cofactor matrix of } A$

$C = [c_{ij}]$ c_{ij} is cofactor of a_{ij}

2×2

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\text{adj}(A) = C^T = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

interchange diagonal elements & change of sign of non diagonal elements

3×3

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$\text{Adj}(A) = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$

Properties

(1)

$A \cdot \text{Adj}(A) = \text{Adj}(A) \cdot A = |A| I_n$

(2)

$|A \text{Adj}(A)| = |A|^{n-1}$

$A_{2 \times 2}$

$|A \text{Adj}(A)| = |A|$

$A_{3 \times 3}$

$|A \text{Adj}(A)| = |A|^2$

③

$$|A \operatorname{adj}(A)| = |A|^{(n-1)^2}$$

$$A_{3 \times 3} \quad |\operatorname{adj}(\operatorname{adj} A)| = |A|^4$$

$$\underbrace{|\operatorname{adj}(\operatorname{adj}(\operatorname{adj} \dots (\operatorname{adj} A) \dots))|}_{K \text{ times}} = |A|^{(n-1)^K}$$

④

$$\operatorname{adj}(kA) = k^{n-1} \operatorname{adj}(A)$$

⑤

$$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$$

$$\text{if } A_{3 \times 3} \Rightarrow \operatorname{adj}(\operatorname{adj}(A)) = |A| A$$

⑥

$$\operatorname{adj}(AB) = \operatorname{adj}(B) \operatorname{adj}(A)$$

⑦

$$\operatorname{adj}(A^{-1}) = (\operatorname{adj}(A))^{-1}$$

⑧

$$\operatorname{adj}(A') = (\operatorname{adj}(A))'$$

⑨

$$A \rightarrow \text{Symmetric} \quad \operatorname{adj}(A) \rightarrow \text{Symmetric}$$

$$A \rightarrow \text{Diagonal} \quad \operatorname{adj}(A) \rightarrow \text{Diagonal}$$

⑩

$$A \rightarrow \text{Singular} \quad \operatorname{adj}(A) \rightarrow \text{Singular}$$

$$A \rightarrow \text{Non singular} \quad \operatorname{adj}(A) \rightarrow \text{Non singular}$$

Inverse of matrix 'A'

B is said to be inverse of matrix A

if

$$A \cdot B = B \cdot A = I$$

$$B = A^{-1}$$

&

$$A = B^{-1} \Rightarrow$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

$$\Rightarrow (A^{-1})^{-1} = A$$

Properties of Inverse

① $|A^{-1}| = \frac{1}{|A|}$

② $(kA)^{-1} = \frac{1}{k} A^{-1}, k \neq 0$ ⑧

$(-A)^{-1} = -A^{-1}$

③ $(A+B)^{-1} \neq A^{-1} + B^{-1}$

④ $(AB)^{-1} = B^{-1} A^{-1}$

$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$

⑤ $(\text{adj}(A))^{-1} = \text{adj}(A^{-1})$

⑥ $(A^{-1})^T = (A^T)^{-1}$

⑦ $A \rightarrow \text{sm} \quad A^{-1} \rightarrow \text{sm}$
 $A \rightarrow \text{diagonal} \quad A^{-1} \rightarrow \text{diagonal}$

$A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}, d_1, d_2, d_3 \neq 0$

$A^{-1} = \begin{bmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{bmatrix}$

Characteristic Equation of Square matrix A_n

→ Characteristic Equation of A_n is $|A_n - \lambda I_n| = 0$

→ Roots of Characteristic equation are known as eigen values

→ Every matrix satisfies its characteristic equation

→ Characteristic equation of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\lambda^2 - (a+d)\lambda + (ad-bc) = 0$$

Hence $A^2 - (a+d)A + (ad-bc)I_2 = 0$